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Интегральное преобразование Лапласа

$$\mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-\lambda t} dt = F(\lambda) \qquad \lambda = a + ib$$

$$\mathcal{L}^{-1}[F(\lambda)] = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(\lambda) e^{\lambda t} d\lambda = f(t)$$

Преобразование Лапласа можно применять если:

1) $f(t)$ определена на $t \in [0, \infty)$
2) $\forall A$ $f(t)$ - кусочно-непрерывная $t \in [0, A]$
3) $\exists M, c, T > 0$, что $|f(t)| \le M e^{ct}$ $\forall t > T$

$\Rightarrow \mathcal{L}[f(t)]$ определено $\forall \operatorname{Re}\lambda = a > c$

Св-ва.

1. Взаимно-однозначное
2. Линейное

   $f(t)$; $g(t)$; a, b - const

   $$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$$

3. Теорема запаздывания

   $$\mathcal{L}[f(t-c) \cdot H(t-c)] = e^{-\lambda c} F(\lambda) \qquad c - \text{const}$$

$$H(t-c) = \begin{cases} 0, & 0 < t < c \\ 1, & t \ge c \end{cases}$$

$$\blacktriangleright \ \mathcal{L}[f(t-c) H(t-c)] = \int_0^\infty f(t-c) H(t-c) e^{-\lambda t} dt =$$

$$= \int_c^\infty f(t-c) e^{-\lambda t} dt = [\xi = t - c] = \int_0^\infty f(\xi) e^{-\lambda \xi - \lambda c} d\xi =$$

$$= \underbrace{\int_0^\infty f(\xi) e^{-\lambda\xi}\cdot e^{-\lambda c}\, d\xi}_{F(\lambda)} = e^{-\lambda c} F(\lambda)$$ ◄

4. Теорема смещения

$\mathcal{L}[e^{ct} f(t)] = F(\lambda - c)$ c - const

► $\mathcal{L}[e^{ct} f(t)] = \int_0^\infty e^{ct} f(t) e^{-\lambda t} dt = \int_0^\infty f(t) e^{-t(\lambda - c)} dt =$

$= F(\lambda - c)$ ◄

5 Свертка

$f(t)$; $g(t)$

$\mathcal{L}[f(t)] = F(\lambda)$; $\mathcal{L}[g(t)] = G(\lambda)$

$\mathcal{L}^{-1}[F(\lambda)\cdot G(\lambda)] = f(t) * g(t)$ - свертка

$$f(t)*g(t) = \int_0^t f(\tau) g(t-\tau) d\tau = \int_0^t g(\tau) f(t-\tau) d\tau$$

Д-ем, что

$\mathcal{L}[f(t)*g(t)] = F(\lambda)\cdot G(\lambda)$

► $\mathcal{L}[f*g] = \int_0^\infty (f*g) e^{-\lambda t} dt = \int_0^\infty e^{-\lambda t}\left[\int_0^t f(\tau) g(t-\tau) d\tau\right] dt =$

$= /*$ $H(t-\tau) = \begin{cases} 1 & \tau \le t \\ 0 & \tau > t \end{cases}$ - зеркально отраженная ф-я Хевисайда */

$$= \int_0^\infty e^{-\lambda t}\left[\int_0^\infty f(\tau) g(t-\tau) H(t-\tau) d\tau\right] dt =$$

$$\int_0^\infty f(\tau)\left[\int_0^\infty g(t-\tau)H(t-\tau)e^{-\lambda t}\,dt\right]d\tau =$$

$\mathcal{L}[g(t-\tau)H(t-\tau)] = e^{-\lambda\tau}G(\lambda)$ Th запаздыв.

$$= \int_0^\infty f(\tau)e^{-\lambda\tau}G(\lambda)\,d\tau = F(\lambda)\cdot G(\lambda)$$

6. преобразование производных

$u(t,x)$

$$\mathcal{L}[u(t,x)] = \int_0^\infty u(t,x)e^{-\lambda t}\,dt = U(\lambda,x)$$

/* $u_{tt} = a^2 u_{xx}$

$u_t = a^2 u_{xx}$ */

u_x:

$$\mathcal{L}[u_x(t,x)] = \int_0^\infty u_x(t,x)e^{-\lambda t}\,dt = \frac{\partial}{\partial x}\left(\int_0^\infty u(t,x)e^{-\lambda t}\,dt\right) =$$

$$= \frac{\partial}{\partial x}U(\lambda,x) = U_x(\lambda,x)$$

$$\mathcal{L}[u_{xx}(t,x)] = \int_0^\infty u_{xx}(t,x)e^{-\lambda t}\,dt = \frac{\partial^2}{\partial x^2}\left(\int_0^\infty u(t,x)e^{-\lambda t}\,dt\right) =$$

$$= U_{xx}(\lambda,x)$$

u_t:

$$\mathcal{L}[u_t(t,x)] = \int_0^\infty u_t(t,x)e^{-\lambda t}\,dt = \int_0^\infty e^{-\lambda t}\,du(t,x) =$$

$$= e^{-\lambda t} u(t,x)\Big|_{t=0}^{t=\infty} - \int_0^\infty u(t,x)\, d e^{-\lambda t} =$$

0 при $t \to \infty$

в силу дост. усл. ∃ инт. преобр Лапласа

$$= -u(0,x) + \lambda\int_0^\infty u(t,x)\, e^{-\lambda t}\, dt = -u(0,x) + \lambda\, U(\lambda,x)$$

$$\mathcal{L}[u_{tt}(t,x)] = \int_0^\infty u_{tt}(t,x)\, e^{-\lambda t}\, dt = \int_0^\infty e^{-\lambda t}\, d u_t(t,x) =$$

$$= e^{-\lambda t} u_t(t,x)\Big|_{t=0}^{t\to\infty} - \int_0^\infty u_t(t,x)\, de^{-\lambda t} =$$

$$= -u_t(0,x) + \lambda\int_0^\infty u_t(t,x)\, e^{-\lambda t}\, dt =$$

$$= -u_t(0,x) + \lambda\int_0^\infty e^{-\lambda t}\, d u(t,x) =$$

$$= -u_t(0,x) + \lambda e^{-\lambda t} u(t,x)\Big|_{t=0}^{t\to\infty} - \lambda\int_0^\infty u(t,x)\, de^{-\lambda t} =$$

$$= -u_t(0,x) - \lambda u(0,x) + \lambda^2\int_0^\infty u(t,x)\, e^{-\lambda t}\, dt =$$

$$= -u_t(0,x) - \lambda u(0,x) + \lambda^2\, U(\lambda,x)$$

Таблица преобразования производных

$$\mathcal{L}[u_x(t,x)] = \bar{U}_x(\lambda, x) \qquad U(\lambda,x) = \int_0^\infty u(t,x)e^{-\lambda t}\,dt$$

$$\mathcal{L}[u_{xx}(t,x)] = \bar{U}_{xx}(\lambda, x)$$

$$\mathcal{L}[u_t(t,x)] = -u(0,x) + \lambda \bar{U}(\lambda,x)$$

$$\mathcal{L}[u_{tt}(t,x)] = -u_t(0,x) - \lambda u(0,x) + \lambda^2 \bar{U}(\lambda,x)$$

Пример

$$u_t = 9u_{xx} \qquad t>0; \quad x>0$$

н.у. $u(0,x) = 0$

г.у. $u(t,0) = 3$

Применяем инт. преобр. Лапласа по t

$$\mathcal{L}[u_t] = \mathcal{L}[9u_{xx}]$$

$$-\underbrace{u(0,x)}_{=0} + \lambda U(\lambda,x) = 9\bar{U}_{xx}(\lambda,x)$$

$$\underline{U_{xx} = \frac{\lambda}{9}U} \quad \text{– О.Д.У}$$

г.у. $\mathcal{L}[u(t,0)] = \mathcal{L}[3\cdot 1]$

$$\underline{U(\lambda,0) = \frac{3}{\lambda}}$$

$$\underline{U(\lambda\to\infty) = 0} \quad \text{доп. усл.}$$

$$\mu^2 = \frac{\lambda}{9} \Rightarrow \mu = \pm\frac{\sqrt{\lambda}}{3}$$

$$U(\lambda,x) = C_1 e^{\frac{\sqrt{\lambda}}{3}x} + C_2 e^{-\frac{\sqrt{\lambda}}{3}x}$$

– общ. реш. з. в образах

$\bar{U}(\lambda \to \infty) = 0 \Rightarrow C_1 \equiv 0$

$\bar{U}(\lambda, 0) = C_2 = \dfrac{3}{\lambda}$

$$U(\lambda, x) = \frac{3}{\lambda} e^{-\frac{\sqrt{\lambda}}{3} x}$$ решение в образах

Действуем на решение обратным преобразованием Лапласа

$f(x)$	$F(\lambda)$
$\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{e^{-a\sqrt{\lambda}}}{\lambda}$

$$\mathcal{L}^{-1}[U(\lambda, x)] = 3 \operatorname{erfc}\left(\frac{x}{6\sqrt{t}}\right) = u(t, x)$$