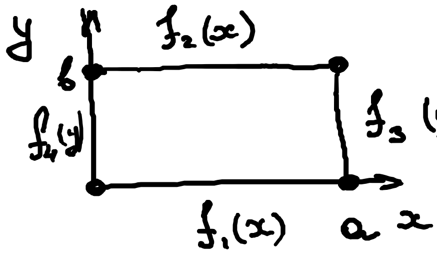
Лекция 13 01.12.2020

$\Delta u = 0$ $\quad 0 < x < a$

$0 < y < b$



$u(x,0) = f_1(x)$

$u(a,y) = f_3(y)$

$u(x,b) = f_2(x)$

$u(0,y) = f_4(y)$

$u(x,y) = u_1(x,y) + Ax + By + Cxy + D$

A, B, C, D - определяем из условий

$(0,0):\quad A\cdot 0 + B\cdot 0 + C\cdot 0 + D = f_1(0) \Rightarrow D = f_1(0)$

$(0,b):\quad A\cdot 0 + B\cdot b + C\cdot 0 + D = f_4(b) \Rightarrow B\cdot b + D = f_4(b)$

$(a,b):\quad A\cdot a + B\cdot b + Cab + D = f_2(a)$

$(a,0):\quad A\cdot a + B\cdot 0 + C\cdot 0 + D = f_3(0) \Rightarrow Aa + D = f_3(0).$

$\Downarrow$

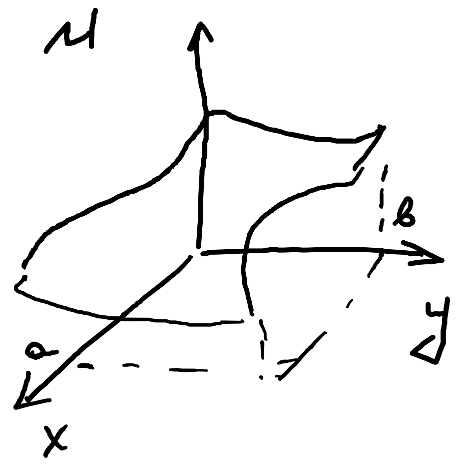
A, B, C, D - известны

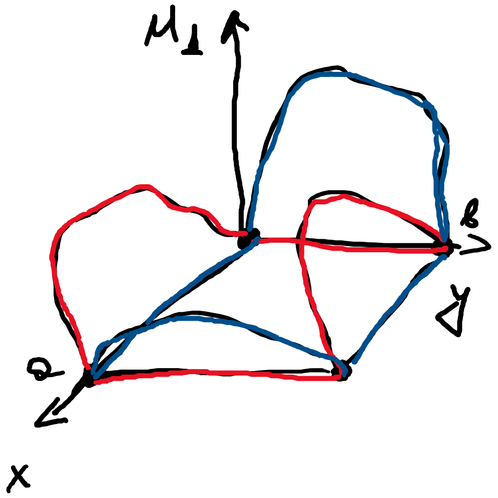
$\Delta u = 0 \Rightarrow \underline{\Delta u_1 = 0}$ $\quad 0 < x < a$

$0 < y < b$

Г.У.

$$u_1(x,0) + Ax + D = f_1(x)$$
$$u_1(a,y) + Aa + By + Cay + D = f_3(y)$$
$$u_1(x,b) + Ax + Bb + Cbx + D = f_2(x)$$
$$u_1(0,y) + By + D = f_4(y)$$





$$u_1(x,y) = u_2(x,y) + u_3(x,y)$$

$$\Delta u_2 = 0 \qquad \begin{array}{l} 0 < x < a \\ 0 < y < b \end{array}$$

$$u_2(x,0) + Ax + D = f_1(x)$$
$$u_2(a,y) = 0$$
$$u_2(x,b) + Ax + Bb + Cbx + D = f_2(x)$$
$$u_2(0,y) = 0$$

$$\Delta u_3 = 0 \qquad \begin{array}{l} 0 < x < a \\ 0 < y < b \end{array}$$

$$u_3(x,0) = 0$$
$$u_3(a,y) + Aa + By + Cay + D = f_3(y)$$
$$u_3(x,b) = 0$$
$$u_3(0,y) + By + D = f_4(y)$$

Задачи для u_2 и u_3 решать умеем

Ответ: $u = u_2 + u_3 + Ax + By + Cxy + D$

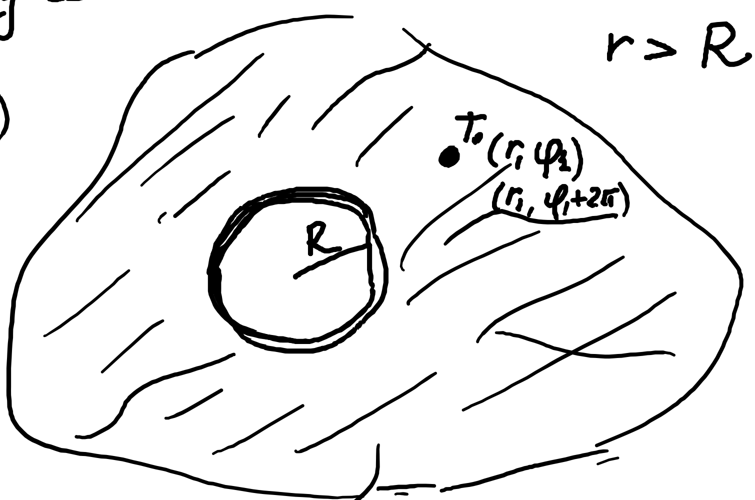
Решение уравнения Лапласа для круга

$r > R$

1)

$0 \le r < R$

2)



$u(r, \varphi)$

$$\left[\begin{array}{l} \Delta u = 0 \\ u(R, \varphi) = g(\varphi) \end{array}\right.$$

1. Внутренняя $\quad 0 \le r < R; \quad \varphi \in [0, 2\pi)$
2. Внешняя $\quad r > R; \quad \varphi \in [0, 2\pi)$.

Важно! Решение задачи – периодическая ф-я с периодом укладывающимся в 2π.

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = 0$$

$$u(r, \varphi) = Y(r)\Phi(\varphi)$$

$$Y''\Phi + \frac{1}{r} Y'\Phi + \frac{1}{r^2} Y\Phi'' = 0 \qquad \Big| : Y\Phi$$

$$\frac{Y''}{Y} + \frac{Y'}{rY} + \frac{\Phi''}{r^2\Phi} = 0 \qquad \Big| \cdot r^2$$

$$\frac{r^2 Y'' + r Y'}{Y} = -\frac{\Phi''}{\Phi} = C$$

$$r^2 Y'' + r Y' = C Y$$

$$\Phi'' = -C\Phi$$

1. $C = -\lambda^2 < 0$

$\Phi'' = \lambda^2 \Phi \Rightarrow \mu^2 = \lambda^2 \Rightarrow \mu = \pm\lambda$

$\Phi = A e^{\lambda\varphi} + B e^{-\lambda\varphi}$ не периодич. ф-я.

$\varnothing$

2. $C = 0$

$\Phi'' = 0 \Rightarrow \Phi = A\varphi + B \Rightarrow A = 0$

$\Phi = B$ - подходит.

3. $C = \lambda^2 > 0$

$\Phi'' = -\lambda^2 \Phi$; $\mu^2 = -\lambda^2 \Rightarrow \mu = \pm i\lambda$

$\Phi = A \sin \lambda\varphi + B \cos \lambda\varphi$

$\lambda = n = 0, 1, 2, \ldots$ Φ - периодич. ф-я с периодом, уклад. в 2π

$\Phi_n(\varphi) = A_n \sin n\varphi + B_n \cos n\varphi$ $n = 0, 1, 2, \ldots$

Решаем задачу для $Y(r)$

$$r^2 Y_n'' + r Y_n' - n^2 Y_n = 0$$

✓ $n \neq 0$

уравнение Эйлера

$$Y_n \sim r^\alpha$$

$$r^2 \alpha(\alpha-1) r^{\alpha-2} + r \alpha r^{\alpha-1} - n^2 r^\alpha = 0$$

$$\alpha^2 - \alpha + \alpha - n^2 = 0$$

$$\alpha^2 = n^2 \Rightarrow \alpha = \pm n$$

$$Y_n = D_n r^n + E_n r^{-n} \qquad n = 1, 2, \ldots$$

$n = 0$

$$r^2 Y_0'' + r Y_0' = 0$$

$$Y_0' = V$$

$$r^2 V' + r V = 0$$

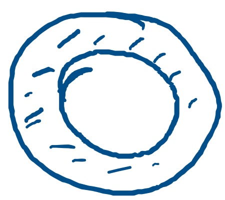
$$\frac{dV}{V} = -\frac{dr}{r}$$

$$\ln V = -\ln r + \tilde{k}$$

$$V = \frac{k}{r}$$

$$Y_0' = \frac{k}{r} \Rightarrow \quad dY_0 = \frac{k\,dr}{r} \Rightarrow Y_0 = k \ln r + F$$

$$Y_0(r) = k \ln r + F$$



Общее решение уравнение Лапласа

$$u(r,\varphi) = \sum_{n=0}^{\infty} Y_n(r)\,\Phi_n(\varphi) = k_0 \ln r + F_0 + \sum_{n=1}^{\infty} \left(D_n r^n + E_n r^{-n}\right) \cdot$$

$$\cdot \left(A_n \sin n\varphi + B_n \cos n\varphi\right)$$

1. Внутренняя задача

$$0 \le r < R$$

$k_0 \equiv 0 \quad E_n \equiv 0$ => ограниченное решение

$$\boxed{u(r,\varphi) = \sum_{n=0}^{\infty} r^n \left(A_n \sin n\varphi + B_n \cos n\varphi \right)}$$

Г.У.

$$u(R,\varphi) = \sum_{n=0}^{\infty} R^n \left(A_n \sin n\varphi + B_n \cos n\varphi \right) = g(\varphi) \quad \Big| \cdot \int_0^{2\pi} \sin k\varphi \, d\varphi$$

$k = 1, 2, \ldots$

$$R^k A_k \pi = \int_0^{2\pi} g(\varphi) \sin k\varphi \, d\varphi$$

$\int_0^{2\pi} d\varphi$, $k=0$

$\int_0^{2\pi} \cos k\varphi \, d\varphi$, $k = 1, 2, \ldots$

$$\boxed{A_k = \frac{1}{\pi R^k} \int_0^{2\pi} g(\varphi) \sin k\varphi \, d\varphi}$$

$k = 1, 2, \ldots$

$$R^k B_k \pi = \int_0^{2\pi} g(\varphi) \cos k\varphi \, d\varphi$$

$$\boxed{B_k = \frac{1}{\pi R^k} \int_0^{2\pi} g(\varphi) \cos k\varphi \, d\varphi}$$

$$B_0 \, 2\pi = \int_0^{2\pi} g(\varphi) \, d\varphi$$

$$\boxed{B_0 = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi) \, d\varphi}$$

2) Внешняя задача

$$r > R$$

Для ограниченного решения при $r \to \infty$

$k_0 \equiv 0$; $D_n \equiv 0$

$$u(r,\varphi) = \sum_{n=0}^{\infty} r^{-n} \left(A_n \sin n\varphi + B_n \cos n\varphi \right)$$

$$u(R,\varphi) = \sum_{n=0}^{\infty} R^{-n} \left(A_n \sin n\varphi + B_n \cos n\varphi \right) = g(\varphi) \quad \Big| \quad \int_0^{2\pi} \sin k\varphi \, d\varphi$$

$k = 1, 2 \ldots$

$$A_k = \frac{R^k}{\pi} \int_0^{2\pi} g(\varphi) \sin k\varphi \, d\varphi$$

$$\int_0^{2\pi} \cos k\varphi \, d\varphi$$

$k = 1, 2, \ldots$

$k = 1, 2, \ldots$

$$B_k = \frac{R^k}{\pi} \int_0^{2\pi} g(\varphi) \cos k\varphi \, d\varphi$$

$$\int_0^{2\pi} d\varphi$$

$k = 1, 2, \ldots$

$$B_0 = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi) \, d\varphi$$