Лекция 16 22.12.2020

Решение эл. уравнений в сферич. сис. коорд.

Задача 1

$\Delta u = 0 \qquad 0 \le r < 5; \quad 0 \le \theta \le \pi; \quad 0 \le \varphi < 2\pi$

$u(5, \theta, \varphi) = 1 - 3\cos^2\theta$

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \alpha_n r^n P_n(\cos\theta)$$

$$u(5, \theta, \varphi) = \sum_{n=0}^{\infty} \alpha_n 5^n P_n(\cos\theta) = 1 - 3\cos^2\theta$$

$$P_2(x) = \frac{3x^2 - 1}{2} \qquad -2P_2(\cos\theta) = 1 - 3\cos^2\theta$$

$\alpha_n \equiv 0 \quad \forall n \neq 2$

$\alpha_2 5^2 = -2 \Rightarrow \alpha_2 = -\frac{2}{25}$

Ответ: $u(r, \theta, \varphi) = -\frac{2}{25} r^2 P_2(\cos\theta)$

Задача 2

$\Delta u = 0 \qquad 0 \le r < 5, \quad 0 \le \theta \le \pi; \quad 0 \le \varphi < 2\pi$

$u(5, \theta, \varphi) = 3 - 2\cos^2\theta$

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \alpha_n r^n P_n(\cos\theta)$$

$$u(5, \theta, \varphi) = \sum_{n=0}^{\infty} \alpha_n 5^n P_n(\cos\theta) = 3 - 2\cos^2\theta$$

$P_2(x) = \frac{3x^2-1}{2}$; $P_1(x) = x$; $P_0(x) = 1$

$$3 - 2\cos^2\theta = -\frac{2}{3}\left(\underbrace{\frac{3\cos^2\theta - 1}{2}}_{P_2(\cos\theta)} + 1\right)\cdot 2 + 3 =$$

$$= -\frac{4}{3}P_2(\cos\theta) - \frac{2}{3} + 3 = -\frac{4}{3}P_2(\cos\theta) + \frac{7}{3}P_0(\cos\theta)$$

$\forall n \neq 0,2 \; \alpha_n \equiv 0$; $\alpha_0 5^0 = \frac{7}{3}$; $\alpha_2 \cdot 5^2 = -\frac{4}{3}$

$$\alpha_0 = \frac{7}{3} \; ; \quad \alpha_2 = -\frac{4}{75}$$

Omsem: $u(r,\theta,\varphi) = \frac{7}{3} - \frac{4}{75} r^2 P_2(\cos\theta)$

Задача 3

$\Delta u = 0 \qquad 0 \le r < 2, \quad 0 \le \theta \le \pi; \quad 0 \le \varphi < 2\pi$

$u(2,\theta,\varphi) = 2 + 3\cos\theta + 4\cos^2\theta + \cos^3\theta$

$$u(r,\theta,\varphi) = \sum_{n=0}^{\infty} \alpha_n r^n P_n(\cos\theta)$$

$$u(2,\theta,\varphi) = \sum_{n=0}^{\infty} \alpha_n 2^n P_n(\cos\theta) = 2 + 3\cos\theta + 4\cos^2\theta + \cos^3\theta$$

$P_3(x) = \frac{5x^3 - 3x}{2}$; $P_2(x) = \frac{3x^2-1}{2}$; $P_1(x) = x$; $P_0(x) = 1$

$$2 + 3\cos\theta + 4\cos^2\theta + \cos^3\theta = \frac{2}{5}\left(\underbrace{\frac{5}{2}\cos^3\theta - \frac{3\cos\theta}{2}}_{P_3} + \frac{3\cos\theta}{2}\right) +$$

$$4\cos^2\theta + 3\cos\theta + 2 = \frac{2}{5}P_3(\cos\theta) + \frac{3}{5}\cos\theta + 3\cos\theta + 2 +$$

$$+ 4\cdot\frac{2}{3}\underbrace{\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)}_{P_2} + \frac{1}{2}\Big) = \frac{2}{5}P_3(\cos\theta) + \frac{8}{3}P_2(\cos\theta) +$$

$$+ \frac{18}{5}\underbrace{\cos\theta}_{P_1} + 2 + \underbrace{\frac{4}{3}}_{P_0} = \frac{2}{5}P_3(\cos\theta) + \frac{8}{3}P_2(\cos\theta) + \frac{18}{5}P_1(\cos\theta)$$

$$+ \frac{10}{3}P_0(\cos\theta)$$

$\alpha_n \equiv 0 \quad \forall n \neq 0, 1, 2, 3$

$\alpha_0 2^0 = \frac{10}{3}$ $\qquad \alpha_0 = \frac{10}{3}$

$\alpha_1 \cdot 2^1 = \frac{18}{5} \Rightarrow \qquad \alpha_1 = \frac{9}{5}$

$\alpha_2 \cdot 2^2 = \frac{8}{3} \qquad \alpha_2 = \frac{2}{3}$

$\alpha_3 \cdot 2^3 = \frac{2}{5} \qquad \alpha_3 = \frac{1}{20}$

Omšem: $u(r,\theta,\varphi) = \frac{10}{3} + \frac{9}{5}rP_1(\cos\theta) + \frac{2}{3}r^2P_2(\cos\theta) + \frac{1}{20}r^3P_3$

Zadaca 4

$\Delta u = 0; \quad 0 \le r < 3; \quad 0 \le \theta \le \pi; \quad 0 \le \varphi < 2\pi$

$u(3,\theta,\varphi) = 4\cos^4\theta + \cos^2\theta$

Omšem: $u(r,\theta,\varphi) = \frac{32}{2835}r^4P_4(\cos\theta) + \frac{62}{189}r^2P_2(\cos\theta) +$

$$+ \frac{17}{15}$$

Задача 5

$\Delta u = 0 \qquad r > 3 \qquad 0 \le \theta \le \pi \qquad 0 \le \varphi < 2\pi$

$u(3, \theta, \varphi) = 3\cos^3\theta + 4$

Ответ $u(r, \theta, \varphi) = \frac{12}{r} + \frac{81}{5r^2} P_1(\cos\theta) + \frac{486}{5r^4} P_3(\cos\theta)$

Д/з

① $\Delta u = 0 \qquad r > 1; \quad 0 \le \theta \le \pi; \quad 0 \le \varphi < 2\pi$

$u(1, \theta, \varphi) = \cos\theta - \cos^2\theta$

② $\Delta u = 0 \qquad 0 \le r < 4, \quad 0 \le \theta \le \pi, \quad 0 \le \varphi < 2\pi$

$u(4, \theta, \varphi) = 1 + 3\cos\theta + 5\cos^2\theta - 8\cos^3\theta$