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Решение неоднородных уравнений для бесконечной струны

$$u_{tt} = a^2 u_{xx} + f(t,x) \qquad t>0 \qquad x\in\mathbb{R} \qquad (1)$$

$u(0,x)=0$
$u_t(0,x)=0$

Сформулируем вспомогательную задачу

$w(t,x,\tau)$

$$w_{tt} = a^2 w_{xx} \qquad t>\tau \qquad x\in\mathbb{R} \qquad (2)$$

$w(t=\tau, x,\tau)=0$
$w_t(t=\tau,x,\tau) = f(\tau,x)$

Докажем, что решение (1) определяется по формуле

$$u(t,x) = \int_0^t w(t,x,\tau)\,d\tau, \text{ где } w \text{ - решение } (2)$$

/* $I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(\alpha,\xi)\,d\xi$

$$I'_\alpha = \int_{a(\alpha)}^{b(\alpha)} f'_\alpha(\alpha,\xi)\,d\xi + f(\alpha,b(\alpha))\,b'_\alpha(\alpha) - f(\alpha,a(\alpha))\,a'_\alpha(\alpha)$$

*/

$$u_{xx} = \int_0^t w_{xx}(t,x,\tau)\,d\tau$$

$$u_t = \int_0^t w_t(t,x,\tau)\,d\tau + \underbrace{w(t,x,\tau=t)}_{=0 \text{ из нач усл } (2)}\cdot 1 - 0 =$$

$$= \int_0^t w_t(t,x,\tau)\,d\tau$$

$$u_{tt} = \int_0^t w_{tt}(t,x,\tau)\,d\tau + \underbrace{w_t(t,x,\tau=t)\cdot 1}_{=f(t,x)} - 0$$

$$= \int_0^t w_{tt}(t,x,\tau)\,d\tau + f(t,x)$$

$$\underbrace{\int_0^t w_{tt}(t,x,\tau)\,d\tau + \cancel{f(t,x)}}_{u_{tt}} = a^2 \cdot \underbrace{\int_0^t w_{xx}(t,x,\tau)\,d\tau}_{u_{xx}} + \cancel{f(t,x)}$$

$$\left[\begin{array}{ll} w_{tt}(t,x,\tau) = a^2 w_{xx}(t,x,\tau) & t>\tau,\quad x\in\mathbb{R} \\ w(t=\tau,x,\tau) = 0 & \\ w_t(t=\tau,x,\tau) = f(\tau,x) & \end{array}\right.$$

$$t^* = t - \tau$$

$$w_{t^*t^*}(t^*,x,\tau) = a^2 w_{xx}(t^*,x,\tau) \qquad t^*>0 \qquad x\in\mathbb{R}$$

$$w(t^*=0,x,\tau) = 0 \qquad = \varphi(x)$$

$$w_{t^*}(t^*=0,x,\tau) = f(\tau,x) \qquad = \psi(x)$$

$$w(t^*,x,\tau) = \frac{1}{2a}\int_{x-at^*}^{x+at^*} f(\tau,\xi)\,d\xi\ ;$$

$$w(t,x,\tau) = \frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau,\xi)\,d\xi$$

$$\boxed{u(t,x) = \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau,\xi)\,d\xi\,d\tau}$$

Усложняем

$$u_{tt} = a^2 u_{xx} + f(t,x) \qquad t>0, \quad x \in R$$

$$u(0,x) = \varphi(x)$$

$$u_t(0,x) = \psi(x)$$

$$u = v + w$$

$$v_{tt} = a^2 v_{xx} \qquad t>0, \ x \in R$$

$$v(0,x) = \varphi(x)$$

$$v_t(0,x) = \psi(x)$$

$$\Downarrow$$

$$v(t,x) = \frac{\varphi(x+at)+\varphi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\,d\xi$$

$$w_{tt} = a^2 w_{xx} + f(t,x) \quad t>0, \ x \in R$$

$$w(0,x) = 0$$

$$w_t(0,x) = 0$$

$$\Downarrow$$

$$w(t,x) = \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau,\xi)\,d\xi\,d\tau$$

Решение ур-я колебаний струны на отрезке

Метод разделения переменных (метод Фурье)

$$u_{tt} = a^2 u_{xx} \qquad t>0, \quad 0<x<l$$

н.у. $u(0,x) = \varphi(x)$

$u_t(0,x) = \psi(x)$

г.у. $u(t,0) = 0$

$u(t,l) = 0$

Решение ищем в виде

$u(t,x) = X(x)\cdot T(t)$ /* ищем нетривиальное решение */

$T''X = a^2 TX''$

н.у $T(0)X(x) = \varphi(x)$

$T'(0)X(x) = \psi(x)$

г.у. $T(t)\cdot X(0) = 0$ $\Rightarrow$ $X(0) = 0$

$T(t)\cdot X(l) = 0$ $\quad X(l) = 0$

ур-е:

$T''X = a^2 TX''$ $\quad |:a^2TX$

$\frac{T''}{a^2T} = \frac{X''}{X} = C$

$t = t_0$ $\quad \forall x \in (0, l)$

$\forall t > 0$ $\quad x = x_0$

$$\begin{cases} X'' \overset{?}{=} CX \\ X(0) = 0 \\ X(l) = 0 \end{cases}$$ - Задача - Штурма - Лиувилля

$T'' = a^2 CT$

1) $C > 0$ $\quad C = \lambda^2$

$X'' = \lambda^2 X$ $\quad \mu^2 = \lambda^2$

$X(0) = 0$ $\quad \mu = \pm\lambda$

$X(l) = 0$ $\quad X = Ae^{\lambda x} + Be^{-\lambda x}$

$$\left.\begin{aligned} X(0) &= A + B = 0 \\ X(l) &= Ae^{\lambda l} + Be^{-\lambda l} = 0 \end{aligned}\right\} \quad \begin{aligned} &A = -B \\ &B(e^{-\lambda l} - e^{\lambda l}) = 0 \Rightarrow B = 0 \\ &\qquad\qquad\qquad\qquad\quad A = 0 \quad \varnothing \end{aligned}$$

2) $C = 0$

$X'' = 0 \Rightarrow X = Ax + B$

$X(0) = 0$

$X(l) = 0$

$X(0) = B = 0$

$X(l) = Al = 0 \Rightarrow A = 0$ $\quad \varnothing$

3) $C < 0 \qquad C = -\lambda^2$

$X'' = -\lambda^2 X \qquad \mu^2 = -\lambda^2$

$X(0) = 0 \qquad \mu = \pm i\lambda$

$X(l) = 0$

$X = A\sin\lambda x + \cancel{B\cos\lambda x}$

$X(0) = B = 0$

$X(l) = A\sin\lambda l = 0$

$\sin\lambda l = 0 \Rightarrow \lambda l = \pi n, \quad n = 1, 2, \ldots$

$\lambda_n = \frac{\pi n}{l}$ - собств. зн-е задачи Ш-Л

$X_n = \underline{\underline{\sin\lambda_n x}}$ - собст. ф-ии з. Ш-Л.

$\{X_n\}$ - образуют ортогональную сист. ф-ий

$$\int_0^l X_n \cdot X_k \, dx = \begin{cases} 0, & n \neq k \\ \frac{l}{2}, & n = k \end{cases}$$

$n \neq k$

$$\int_0^l \sin\lambda_n x \cdot \sin\lambda_k x \, dx = \int_0^l \sin\frac{\pi n x}{l} \sin\frac{\pi k x}{l} \, dx =$$

$$= /* \ \sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \ */ =$$

$$= \frac{1}{2}\int_0^l \cos\frac{\pi x}{l}(n-k) - \cos\frac{\pi x}{l}(n+k) \, dx =$$

$$= \frac{1}{2}\left[\frac{l}{\pi(n-k)} \sin \frac{\pi x}{l}(n-k)\Big|_0^l - \frac{l}{\pi(n+k)} \sin \frac{\pi x}{l}(n+k)\Big|_0^l\right] =$$

$$= \frac{1}{2}\left[\frac{l}{\pi(n-k)} \underbrace{\sin \pi(n-k)}_{=0} - \frac{l}{\pi(n+k)} \underbrace{\sin \pi(n+k)}_{=0}\right] = 0$$

$n=k$

$$\int_0^l \sin^2 \lambda_k x \, dx = \int_0^l \sin^2 \frac{\pi k x}{l} \, dx = /* \sin^2 \alpha = \frac{1}{2}(1-\cos 2\alpha) */$$

$$= \frac{1}{2}\int_0^l 1 - \cos \frac{2\pi k x}{l} \, dx = \frac{1}{2}\left[x\Big|_0^l - \frac{l}{2\pi k} \sin \frac{2\pi k x}{l}\Big|_0^l \right] =$$

$$= \frac{1}{2}\left[l - \frac{l}{2\pi k} \overbrace{\sin \frac{2\pi k \cancel{l}}{\cancel{l}}}^{=0} \right] = \frac{l}{2}$$