Лекция № 6 13.10.2020

$$u_{tt} = a^2 u_{xx} \qquad t>0; \quad x \in (0,\ell)$$

н.у. $u(0,x) = \varphi(x)$

$u_t(0,x) = \psi(x)$

г.у $u(t,0) = 0$

$u(t,\ell) = 0$

$$u(t,x) = T(t)X(x)$$

$X'' = CX$, $C = -\lambda^2$ $\quad X_n(x) = \sin\lambda_n x$ ✓

$X(0) = 0$ $\Rightarrow$ $\lambda_n = \dfrac{\pi n}{\ell}$ $\quad n = 1, 2 \ldots$

$X(\ell) = 0$

$$\int_0^\ell X_k(x) X_n(x)\,dx = \begin{cases} 0 & n \neq k \\ \dfrac{\ell}{2} & n = k \end{cases}$$

$$T'' = a^2 C T \Rightarrow T_n'' = -a^2\lambda_n^2 T_n$$

$$\mu_n^2 = -a^2\lambda_n^2 \Rightarrow \mu_n = \pm i a\lambda_n$$

$$T_n = D_n \sin a\lambda_n t + E_n \cos a\lambda_n t \qquad ✓$$

$$\underline{u(t,x)} = \sum_{n=1}^\infty T_n(t) X_n(x) = \underline{\sum_{n=1}^\infty (D_n \sin a\lambda_n t + E_n \cos a\lambda_n t)\cdot}$$

$\underline{\cdot \sin\lambda_n x}$ – общее решение исходной задачи

н.у.

$$u(0,x) = \sum_{n=1}^\infty E_n \sin\lambda_n x = \varphi(x) \qquad \Big|\int_0^\ell \sin\lambda_k x\,dx$$

$$\int_0^\ell \sum_{n=1}^\infty E_n \sin\lambda_n x \cdot \sin\lambda_k x\,dx = \int_0^\ell \varphi(x)\sin\lambda_k x\,dx$$

$$\mathcal{E}_k = \frac{2}{\ell}\int_0^\ell \varphi(x)\sin\lambda_k x\,dx \qquad k=1,2,\dots$$

$$u_t(0,x)=\psi(x)$$

$$u_t(t,x)=\sum_{n=1}^{\infty}\left(\mathcal{D}_n a\lambda_n \cos a\lambda_n t-\mathcal{E}_n a\lambda_n \sin a\lambda_n t\right)\sin\lambda_n x$$

$$u_t(0,x)=\sum_{n=1}^{\infty}\mathcal{D}_n a\lambda_n \sin\lambda_n x=\psi(x) \qquad \Big|\int_0^\ell \sin\lambda_k x\,dx$$

$$\sum_{n=1}^{\infty}\int_0^\ell \mathcal{D}_n a\lambda_n \sin\lambda_n x\cdot\sin\lambda_k x\,dx=\int_0^\ell\psi(x)\sin\lambda_k x\,dx$$

$$\mathcal{D}_k a\lambda_k\frac{\ell}{2}=\int_0^\ell\psi(x)\sin\lambda_k x\,dx \qquad k=1,2,\dots$$

$$\mathcal{D}_k=\frac{2}{a\lambda_k\ell}\int_0^\ell\psi(x)\sin\lambda_k x\,dx$$

Та же задача, только на границе усл. II рода

$$u_{tt}=a^2u_{xx} \qquad t>0;\quad x\in(0,\ell)$$

н.у. $u(0,x)=\varphi(x)$

$u_t(0,x)=\psi(x)$

г.у. $u_x(t,0)=0$

$u_x(t,\ell)=0$

$$u(t,x)=T(t)\cdot X(x)$$

$$T''X=a^2TX'' \qquad |:a^2TX$$

$$\frac{T''}{a^2T}=\frac{X''}{X}=C$$

$t=t_0 \qquad \forall x\in(0,\ell)$

$\forall t>0 \qquad x=x_0$

$X'' = CX$ $\qquad\qquad$ $T'' = a^2 C T$

г.у. $T(t)X'(0)=0$

$T(t)X'(l)=0$

$\Downarrow$

$X'(0)=0$

$X'(l)=0$

$X'' = CX$

$X'(0)=0$ $\quad$ – З. Ш-Л.

$X'(l)=0$

1) $C=\lambda^2>0$

$X''=\lambda^2 X \qquad \mu^2=\lambda^2 \Rightarrow \mu \pm \lambda$

$X(x)=Ae^{\lambda x}+Be^{-\lambda x}$

$X'(x)=A\lambda e^{\lambda x}+B(-\lambda)e^{-\lambda x}$

$X'(0)=A\lambda - B\lambda = 0 \Rightarrow A=B$

$X'(l)=A\lambda e^{\lambda l} - A\lambda e^{-\lambda l}=0 \Rightarrow A=0 \Rightarrow B=0 \quad \varnothing$

2) $C=0$

$X''=0 \Rightarrow X=Ax+B$

$X'(x)=A$

$X'(0)=A=0$

$X'(l)=A=0$

$X=B$ – подходит

3) $C=-\lambda^2<0$

$X''=-\lambda^2 X \Rightarrow \mu^2=-\lambda^2 \Rightarrow \mu=\pm i\lambda$

$X(x) = A\sin\lambda x + B\cos\lambda x$ $\qquad X_n = \cos\lambda_n x$

$X'(x) = A\lambda\cos\lambda x - B\lambda\sin\lambda x$

$X'(0) = A\lambda = 0 \Rightarrow A = 0$

$X'(l) = -B\lambda\sin\lambda l = 0 \Rightarrow \sin\lambda l = 0$

$\lambda_n = \frac{n\pi}{l} \qquad n = 0, 1, 2, \ldots$

$$\underline{X_n = \cos\lambda_n x}$$

$\lambda_n = \frac{\pi n}{l} \qquad n = 0, 1, 2 \ldots$

Решаем з. для T

$T_n'' = -a^2\lambda_n^2 T_n$

$\mu_n^2 = -a^2\lambda_n^2 \Rightarrow \mu_n = \pm i a\lambda_n$

$$\underline{T_n = D_n\sin a\lambda_n t + C_n\cos a\lambda_n t}$$

$u(t,x) = \sum_{n=0}^{\infty} T_n(t)X_n(x) =$

$= \sum_{n=0}^{\infty}(D_n\sin a\lambda_n t + C_n\cos a\lambda_n t)\cos\lambda_n x$ ✓

Н.У. $u(0,x) = \varphi(x)$

$u(0,x) = \sum_{n=0}^{\infty} C_n\underline{\cos\lambda_n x} = \varphi(x) \quad \Big| \int_0^l \cos\lambda_k x\, dx$

$C_k \frac{l}{2} = \int_0^l \varphi(x)\cos\lambda_k x\, dx \qquad k = 0, 1, \ldots$

$C_k = \frac{2}{l}\int_0^l \varphi(x)\cos\lambda_k x\, dx$ ✓

$u_t(0,x) = \psi(x)$

$$u_t(0,x) = \sum_{n=0}^{\infty} D_n a \lambda_n \cos \lambda_n x = \psi(x) \qquad \Big| \int_0^l \cos\lambda_k x \, dx$$

$$D_k a \lambda_k \frac{l}{2} = \int_0^l \psi(x) \cos \lambda_k x \, dx$$

$$D_k = \frac{2}{a\lambda_k l} \int_0^l \psi(x) \cos \lambda_k x \, dx \qquad \checkmark$$

Решения з. Штурма - Лиувилля

$X_n'' = -\lambda_n^2 X_n \qquad x \in (0, l)$

ГГ \ ПГ	I рода	II рода
I рода	$X_n = \sin \lambda_n x$ $\lambda_n = \frac{\pi n}{l}$; $n = 1, 2, \ldots$	?
II рода	$X_n = \cos \lambda_n x$ $\lambda_n = \frac{\pi}{2l} + \frac{\pi n}{l}$, $n = 0, 1, 2 \ldots$	$X_n = \cos \lambda_n x$ $\lambda_n = \frac{\pi n}{l}$ $\quad n = 0, 1, \ldots$

$u_{tt} = a^2 u_{xx} \qquad t > 0 \quad x \in (0, l)$

н.у. $u(0, x) = \varphi(x)$

$u_t(0, x) = \psi(x)$

г.у. $u_x(t, 0) = 0$

$u(t, l) = 0$